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(1 + 1)-dimensional Hamiltonian systems as symmetry constraints of the Kadomtsev–Petviashvili equation

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Abstract. In this paper we consider linear problems associated with the Kadomtsev–Petviashvili equation. We prove that the linear problems are (1+1)-dimensional Hamiltonian systems under the symmetry constraints. Moreover, we find that the Hamiltonian flows of the linear problems are commutative.

1. Introduction

It has been shown in [1–8] that the linear system of a integrable (1+1)-dimensional system can be constrained to the integrable system of finite dimension (i.e. 0+1 dimension). Recently the above study has also been generalized to soliton equations in two spatial and one temporal (i.e. 2+1) dimensions [9–12]. The main reasons for this research are that on the one hand, some kinds of solutions of a $(d+1)$-dimensional ($d=1$ or 2) system can be obtained by solving the d-dimensional systems reduced from it, and on the other, the constraint offers a way to study the properties of a d-dimension of system from those of a $(d+1)$-dimensional one.

In [9], Yi Cheng and Yishen Li considered the linear problems

$$\psi_y = \psi_{xx} + 2u\psi \tag{1.1a}$$

$$\psi^*_y = -\psi^*_{xx} - 2u\psi^* \tag{1.1b}$$

and

$$\psi_t = \psi_{xxx} + 3u\psi_x + \tfrac{3}{2}u_x\psi - \tfrac{3}{2}(D^{-1}u_y)\psi \tag{1.2a}$$

$$\psi^*_t = \psi^*_{xxx} + 3u\psi^*_x + \tfrac{3}{2}u_x\psi^* - \tfrac{3}{2}(D^{-1}u_y)\psi^* \tag{1.2b}$$

associated with the Kadomtsev–Petviashvili (KP) equation

$$u_t = \tfrac{1}{4}u_{xxx} + 3uu_x + \tfrac{3}{4}D^{-1}u_{yy}. \tag{1.3}$$

Under the constraint condition

$$K_0 \equiv u_x = (\psi\psi^*)_x \tag{1.4}$$

they prove that the systems of (1.1) and (1.2) are the second and third equations of the AKNS hierarchy respectively, and they get a new kind of solution of the KP equation.

Konopelchenko *et al* [10] proved that the system (1.1) is the $(1+1)$-dimensional Hamiltonian system under the constraint conditions

$$K_n = k(\psi\psi^*)_x \qquad n = 0, 1, 2$$

where K_n are the first three symmetries of the KP equation. However, they have not proved that the system (1.1) is the $(1+1)$-dimensional Hamiltonian system under the general symmetry constraint condition

$$K_n = k(\psi\psi^*)_x \qquad n \in \mathbb{N} \cup \{0\}. \tag{1.5}$$

Also they have no ideas to prove that the system (1.2) is $(1+1)$-dimensional Hamiltonian system.

The purpose of this paper is to prove that systems (1.1) and (1.2) are the $(1+1)$-dimensional Hamiltonian systems by any higher-order symmetry constraint condition of (1.5) and their Hamiltonian flows are commutative in pairs. The deeper results, such as a common infinite set of involutive integrals, can be obtained by using the Adler–Kostant–Symes theorem, which will be left to the sequel [12].

For convenience, in section 2, a brief introduction to Sato's theory will be given. In section 3, we prove that systems (1.1) and (1.2) are the $1+1$-dimensional Hamiltonian systems by any higher-order symmetry constraint. Finally, in section 4, we prove that the Hamiltonian flows of (1.1) and (1.2) are commutative in pairs.

2. Sato's theory [13–15]

In this section, we shall describe the framework of Sato's theory. The notions of n-reduction and n-reduction stationary Sato's equation (n-RSSE) will be introduced. Moreover, the Hamiltonian formulation for n-RSSE will be given.

Let us introduce a microdifferential operator

$$\mathrm{L} = \sum_{i=-\infty}^{1} u_i \mathrm{D}^i = \mathrm{D} + u_{-1}\mathrm{D}^{-1} + u_{-2}\mathrm{D}^{-2} + \ldots \tag{2.1}$$

where we assume $u_1 = 1$, $u_0 = 0$ and $u_{-1}, u_{-2}, \ldots$ are to be functions of an infinite set of variables $t = (x, t_2, t_3, \ldots)$. The operator $\mathrm{D} = \mathrm{d}/\mathrm{d}x$ is the usual differential operator, its inverse powers $\mathrm{D}^{-1}, \mathrm{D}^{-2}, \ldots$ may be regarded as formal integration operators acting via the Leibniz rule

$$\mathrm{D}^i f = \sum_{j=0}^{\infty} \binom{1}{j} f^{(j)} \mathrm{D}^{i-j}. \tag{2.2}$$

Here, f is the multiplication operator given by a function $f(t)$ and $f^{(j)} = (\mathrm{d}^j/dx^j)f$ with

$$\binom{i}{j} = \frac{i(i-1)\ldots(i-j+1)}{j!}.$$

The coefficients in (2.2) are defined for arbitrary integers i such that, for negative i, the application of the integration D^i is defined as infinite expansion (2.2) into negative power of D.

For any formal pseudodifferential operator $\mathrm{X} = \Sigma_{-\infty}^{n} \mathrm{X}_i \mathrm{D}^i$ we split this operator into its positive and its negative parts defined by

$$\mathrm{X}_+ = \sum_{i=0}^{n} \mathrm{X}_i \mathrm{D}^i \qquad \text{and} \qquad \mathrm{X}_- = \sum_{i=-\infty}^{-1} \mathrm{X}_i \mathrm{D}_i$$

respectively.

We can easily get the nonlinear evolution equation

$$\frac{d}{dt_n}L_+^m - \frac{d}{dt_m}L_+^n = [L_+^n, L_+^m] \tag{2.3}$$

from the Lax equation

$$\frac{d}{dt_m}L = [L_+^m, L] = [L, L_-^m] \tag{2.4}$$

Now, we introduce the following notation which will be useful in later discussion.

Definition 1. We state L to be n-reduction if there exists a least natural number n such that $L_+^n = L^n$.

Definition 2. The equation

$$\frac{d}{dt_m}L^n = [L_+^m, L^n] = [L^n, L_-^m] \tag{2.4}$$

is called an n-RSSE if L has n-reduction.

Example 1. If L has 2-reduction, i.e. [16]

$$\begin{aligned} L &= (D^2 + 2u_{-1})^{1/2} \\ &= D + u_{-1}D^{-1} - \tfrac{1}{2}u_{-1}^{(1)}D^{-2} + \tfrac{1}{4}(u_{-1}^{(2)}D^{-2} + \tfrac{1}{4}(u_{-1}^{(2)} - 2u_{-1}^2)D^{-3} + \ldots \end{aligned}$$

equation (2.4) reads as the KdV hierarchy ($m = 2k+1$). In particular ($m = 3$), (2.4) reads as the KdV equation

$$u_{t_3} = \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x. \tag{2.5}$$

Example 2. If L has 3-reduction, i.e. [16]

$$\begin{aligned} L &= (D^3 + 3u_{-1}D + 3u_{-1}^{(1)} + 3u_{-2})^{1/3} \\ &= D + u_{-1}D^{-1} + u_{-2}D^{-2} + (-\tfrac{1}{2}u_{-1}^{(2)} - u_{-2}^{(1)} + u_{-1}^2)D^{-3} + \ldots \end{aligned}$$

equation (2.4) becomes the Boussinesq hierarchy. Moverover equation (2.4) becomes the Boussinesq equation

$$\tfrac{3}{4}u_{t_2t_2} + (\tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x)_x = 0 \tag{2.6}$$

if $m = 2$.

In order to obtain the Hamiltonian formulation for equation (2.4), we recall the results of Adler [17]. In his paper, Adler proved that the n-RSSE (2.4) is an integrable Hamiltonian system in 1+1-dimensions, and he also gave the involutive conserved laws

$$H_{m+n} = \frac{n}{m+n}\operatorname{tr} L^{m+n}.$$

Here the trace of a pseudodifferential operator was introduced as the integration of the coefficient of D^{-1}, i.e.

$$\operatorname{tr}(\ldots + a_{-2}D^{-2} + a_{-1}D^{-1} + a_0 + a_1D + \ldots) = a_{-1}.$$

However, in this paper, we use the method of Mumford [18] to find the following results which are similar to the results of [19].

Theorem 1. Let

$$\mathrm{L}^n=\sum_{j=0}^{n}\alpha_{j,n}\mathrm{D}^j \qquad \alpha_{n,n}=1 \qquad \alpha_{n-1,n}=0 \qquad \text{and} \qquad \mathrm{L}^m=\sum_{i=1}^{\infty}\mathrm{D}^{-i}\beta_{-i,m}$$

*then n-*RSSE (2.4) *can be written as the following Hamiltonian system.*

$$\alpha_{n_{t_m}}=A_{n,n}\frac{\delta H_{m+n}}{\delta\alpha_n} \tag{2.7}$$

where

$$\alpha_n=(\alpha_{0,n},\alpha_{1,n},\ldots,\alpha_{n-2,n})^T \qquad \frac{\delta}{\delta\alpha_n}=\left(\frac{\delta}{\delta\alpha_{0,n}},\frac{\delta}{\delta\alpha_{1,n}},\ldots,\frac{\delta}{\delta\alpha_{n-2,n}}\right)^T$$

$$H_{m+n}=\frac{n}{m+n}\operatorname{tr}\mathrm{L}^{m+n} \qquad \frac{\delta}{\delta u}\equiv\sum_{k\geq 0}(-\mathrm{D})^k\frac{\partial}{\partial u^{(k)}}$$

is the variational derivative [17–19]

$$A_{n,n}=\begin{pmatrix} A_{n,1,n} & A_{n,2,n} & \cdots & A_{n,n-1,n} \\ A_{n-1,1,n} & \cdots & A_{n-1,n-2,n} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{3,1,n} & A_{3,2,n} & 0 & 0 \\ A_{2,1,n} & 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

and

$$A_{h,i,n}=\sum_{j=0}^{n-2}\left[\binom{n-i-j}{h-i-j}\alpha_{n-j,n}\mathrm{D}^{h-i-j}-\binom{-i}{h-i-j}\mathrm{D}^{h-i-j}\alpha_{n-j,n}\right]. \tag{2.9}$$

Furthermore, equation (2.7) has a common infinite set of conserved densities $H_{l+n}(l\in\mathbb{N})$ which are in involution in pairs

$$\{H,G\}\equiv\operatorname{tr}(\mathrm{L}^n[\mathrm{d}H,\mathrm{d}G])=\sum_{j=0}^{n-2}\frac{\delta H}{\delta\alpha_{j,n}}\left(A_{n,n}\frac{\delta G}{\delta\alpha_n}\right)_j\sim 0 \tag{2.10}$$

where the notation $f\sim 0$ means that $f=\mathrm{D}g$ for some g and $(\cdot)_j$ the jth element of $(\cdot)$.

3. Hamiltonian systems

The aim of this section is to prove that the linear problems (1.1) and (1.2) are the Hamiltonian systems in 1+1-dimensions under the constraint conditions (1.5).

As we know, the x-derivative of the squared eigenfunction $\psi\psi^*$ is the generating function for symmetries of the KP equation, i.e. the function $(\psi\psi^*)_x$ is the symmetry of the KP equation and it can deduce a series of symmetries of the KP equation if ψ and ψ^* satisfy (1.1) and (1.2) respectively [10, 20–22].

Now we choose $m=2$ in (2.3); we can obtain a series of symmetries

$$K_{n-1}=\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t_2}L_+^n+[\mathrm{L}_+^n,\mathrm{L}_+^2]\right) \qquad n=1,2,\ldots.$$

We consider the linear system (A)

$$\text{(A)}\qquad \begin{aligned} &\psi_{t_2}=\psi_{xx}+2u_{-1}\psi && (1.1a)\\ &\psi^*_{t_2}=-\psi^*_{xx}-2u_{-1}\psi^* \qquad u_{-1}=u && (1.1b)\end{aligned}$$

$$K_{n-1}\equiv\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t_2}L_+^n+[\mathrm{L}_+^n,\mathrm{L}_+^2]\right)=(\psi\psi^*)_x \tag{3.1}$$

for fixed n. For system (A) we have the following theorem.

Theorem 2. System (A) for fixed n is a (1+1)-dimensional Hamiltonian system which can be written as

$$\begin{pmatrix}\psi\\ \alpha_{0,n}\\ \vdots\\ \alpha_{n-2,n}\\ \psi^*\end{pmatrix}_{t_2}=\begin{pmatrix}0&0&\dots&0&1\\ 0&&&&0\\ \vdots&&A_{n,n}&&\vdots\\ 0&&&&0\\ -1&0&\dots&0&0\end{pmatrix}\frac{\delta\tilde{H}_{2,n}}{\delta(\psi,\alpha_n,\psi^*)} \tag{3.2}$$

where the conserved density

$$\tilde{H}_{2,n}=H_{2+n}+\left(\psi_{xx}+\frac{2}{n}\alpha_{n-2,n}\psi\right)\psi^*$$

$$H_{2+n}=\frac{n}{2+n}\operatorname{tr}(\mathrm{L}_+^n)^{(2+n)/n}$$

and

$$\frac{\delta}{\delta(\psi,\alpha_n,\psi^*)}=\left(\frac{\delta}{\delta\psi},\frac{\delta}{\delta\alpha_{0,n}},\dots,\frac{\delta}{\delta\alpha_{n-2,n}},\frac{\delta}{\delta\psi^*}\right)^T.$$

Moreover, the Poisson bracket is defined by

$$\{\tilde{H},\tilde{G}\}=\sum_{j=0}^{n}\left(\frac{\delta\tilde{H}}{\delta(\psi,\alpha_n,\psi^*)}\right)_j\left(B_{n,n}\frac{\delta\tilde{G}}{\delta(\psi,\alpha_n,\psi^*)}\right)_j \tag{3.3}$$

for the conserved densities $\tilde{H}$ and $\tilde{G}$, here

$$B_{n,n}=\begin{pmatrix}0&0&\dots&0&1\\ 0&&&&0\\ \vdots&&A_{n,n}&&\vdots\\ 0&&&&0\\ -1&0&\dots&0&0\end{pmatrix}. \tag{3.4}$$

Proof. From (3.1) we get

$$\frac{\mathrm{d}}{\mathrm{d}t_2}\mathrm{L}_+^n=[\mathrm{L}_+^2,\mathrm{L}_+^n]+2(\psi\psi^*)_x. \tag{3.5}$$

Using theorem 1, we obtain

$$\alpha_{n_{t_2}}=A_{n,n}\frac{\delta H_{2+n}}{\delta\alpha_n}+\begin{pmatrix}2(\psi\psi^*)_x\\ 0\\ \vdots\\ 0\end{pmatrix} \tag{3.6}$$

where $H_{2+n}=n/(2+n)\operatorname{tr}(L_+^n)^{(n+2)/n}$.

To take (2.9) and (2.1) into account, we find $A_{i,i-1,n} = n\mathrm{D}$ and $u_{-1} = (1/n)\alpha_{n-2,n}$ respectively, thus system (A) can be rewritten as

$$\begin{pmatrix} \phi \\ \alpha_{0,n} \\ \vdots \\ \alpha_{n-2,n} \\ \psi^* \end{pmatrix}_{t_2} = B_{n,n} \frac{\delta \tilde{H}_{2,n}}{\delta(\psi, \alpha_n, \psi^*)}$$

where

$$\tilde{H}_{2,n} = H_{2+n} + \left(\psi_{xx} + \frac{2}{n} \alpha_{n-2,n} \psi \right) \psi^*.$$

Finally by using (2.10) we easily get (3.3). This ends the proof of the theorem. □

Examples. As the first and simplest example, we choose the constraint condition $K_0 \equiv u_x = (\psi\psi^*)_x$, then (3.2) becomes as

$$\begin{pmatrix} \psi \\ \psi^* \end{pmatrix}_{t_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\delta \tilde{H}_{2,1}}{\delta(\psi, \psi^*)} \tag{3.7}$$

where $\tilde{H}_{2,1} = (\psi_{xx} + 2\psi\psi^*)\psi^*$. This is the second equation of the AKNS hierarchy.

Next, we choose the constraint $K_1 = u_{t_2} = (\psi\psi^*)_x$, then (3.2) yields

$$\begin{pmatrix} \psi \\ \alpha_{0,2} \\ \psi^* \end{pmatrix}_{t_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2\mathrm{D} & 0 \\ -1 & 0 & 0 \end{pmatrix} \frac{\delta \tilde{H}_{2,2}}{\delta(\psi, \alpha_{0,2}, \psi^*)} \tag{3.8}$$

where $\tilde{H}_{2,2} = (\psi_{xx} + \alpha_{0,2}\psi)\psi^*$. This system is closely connected with the Yajima-Oikawa system [23].

Now we consider system (B)

(B)

$$\psi_{t_3} = \psi_{xxx} + 3u_{-1}\psi_x + \tfrac{3}{2}u_{-1x}\psi - \tfrac{3}{2}(\mathrm{D}^{-1}u_{-1y})\psi \tag{3.9a}$$

$$\psi^*_{t_3} = \psi^*_{xxx} + 3u_{-1}\psi^*_x + \tfrac{3}{2}u_{-1x}\psi^* - \tfrac{3}{2}(\mathrm{D}^{-1}u_{-1y})\psi^* \tag{3.9b}$$

$$K_{n-1} \equiv \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t_2} \mathrm{L}^n_+ + [\mathrm{L}^n_+, \mathrm{L}^2_+] \right) = (\psi\psi^*)_x \tag{3.1}$$

for the same fixed n.

First, we give the following lemmas which will be useful in later discussion.

Lemma 1. The constraint condition (3.1) is equivalent to the following constraint condition:

$$\frac{\mathrm{d}}{\mathrm{d}t_3} \mathrm{L}^n_+ + [\mathrm{L}^n_+, \mathrm{L}^3_+] = 3(\psi\psi^*)_x \mathrm{D} + 3(\psi_x\psi^*)_x. \tag{3.10}$$

Proof. (i) We assume that (3.1) is true; let us denote

$$A = \frac{\mathrm{d}}{\mathrm{d}t_3} \mathrm{L}^n_+ + [\mathrm{L}^n_+, \mathrm{L}^3_+] \tag{3.11}$$

and suppose $A=\Sigma_{i=0}^{n} A_i D^i$. Here $A_i (i=0, 1, \ldots, n)$ are the functions. Hence we get

$$\frac{d}{dt_2} L_+^n = [L_+^2, L_+^n] + 2K_{n-1} \qquad (2K_{n-1} = 2(\psi\psi^*)_x) \tag{3.12}$$

$$\frac{d}{dt_3} L_+^n = [L_+^3, L_+^n] + A. \tag{3.13}$$

Using the Jacobian identity, the KP equation and (3.12) and (3.13), we find

$$[L_+^2, A] + 2K_{n-1_{t_3}} = [L_+^3, 2K_{n-1}] + A_{t_2}. \tag{3.14}$$

Now inserting $L_+^2 = D^2 + 2u_{-1}$, $L_+^3 = D^3 + 3u_{-1}D + \frac{3}{2}u_{-1}^{(1)} + \frac{3}{2}(D^{-1}u_{-1_{yy}})$ and $A = \Sigma_{i=0}^{n} A_i D^i$ into (3.14), we obtain

$$A_i = 0 \qquad (i = 2, 3, \ldots, n) \tag{3.15}$$

$$2A_1^{(1)} - 6K_{n-1}^{(1)} = 0 \tag{3.16}$$

$$A_1^{(2)} + 2A_0^{(1)} - 6K_{n-1}^{(2)} = A_{1_{t_2}} \tag{3.17}$$

$$A_0^{(2)} - 2A_1 u_{-1}^{(u)} - 2K_{n-1}^{(3)} - 6u_{-1}K_{-1}^{(1)} = A_{0_{t_2}} - K_{n-1_{t_3}}. \tag{3.18}$$

Solving the equations (3.15)-(3.19), we finally get

$$A_i = 0 \qquad (i = 2, \ldots, n) \tag{3.15}$$

$$A_1 = 3(\psi\psi^*)_x \tag{3.19}$$

$$A_0 = 3(\psi_x\psi^*)_x. \tag{3.20}$$

This proves lemma 1 on one hand.

(ii) On the other hand, if we suppose that (3.10) is true, condition (3.1) can be obtained by using a similar method.

This ends the proof of lemma 1. □

Lemma 2. System (B) can be written as system (C):

$$\psi_t = \psi_{xxx} + \frac{3}{n}\alpha_{n-2,n}\psi_x + \frac{3(3-n)}{2n}\alpha_{n-2,n}^{(1)}\psi + \frac{3}{n}\alpha_{n-3,n}\psi \tag{3.21a}$$

$$\psi_t^* = \psi_{xxx}^* + \frac{3}{n}\alpha_{n-2,n}\psi_x^* + \frac{3(3-n)}{2n}\alpha_{n-2,n}^{(1)}\psi^* - \frac{3}{n}\alpha_{n-3,n}\psi \tag{3.21b}$$

$$\frac{d}{dt_3} L_+^n + [L_+^n, L_+^3] = 3(\psi\psi^*)_x D + 3(\psi_x\psi^*)_x. \tag{3.10}$$

Proof. Let us put $L_+^n = \Sigma_{i=0}^{n} \alpha_{i,n} D^i$ and $L_+^2 = D^2 + (2/n)\alpha_{n-2,n}$ into (3.1); we find

$$\alpha_{n-2,n_{t_2}} = 2\alpha_{n-3,n}^{(1)} + (2-n)\alpha_{n-2,n}^{(2)}$$

i.e.

$$D^{-1}u_{-1_{t_2}}=\frac{2}{n}\alpha_{n-3,n}+\frac{2-n}{n}\alpha^{(1)}_{n-2,n} \tag{3.22}$$

and taking (1.2) into account we easily obtain the result. This ends the proof. □

Finally, we prove the last of the theorems in this section.

Theorem 3. System (B) for the same n is a (1+1)-dimensional Hamiltonian system which reads as

$$\begin{pmatrix}\psi\\ \alpha_n\\ \psi^*\end{pmatrix}_{t_3}=B_{n,n}\frac{\delta\tilde{H}_{3,n}}{\delta(\psi,\alpha_n,\psi^*)} \tag{3.23}$$

where

$$\tilde{H}_{3,n}=H_{3+n}+\left(\psi_{xxx}+\frac{3}{n}\alpha_{n-2,n}\psi_x+\frac{3(3-n)}{2n}\alpha^{(1)}_{n-2,n}\psi+\frac{3}{n}\alpha_{n-3,n}\psi\right)\psi^* \tag{3.24}$$

$$H_{3+n}=\frac{n}{3+n}\operatorname{tr}(L_+^n)^{(3+n)/n}$$

and the Poisson bracket is the same as (3.3).

Proof. First, as a result of direct calculus of variations [17, 24], we find

$$\frac{\delta\tilde{H}_{3,n}}{\delta\alpha_{n-2,n}}=\frac{\delta H_{3+n}}{\delta\alpha_{n-2,n}}+\frac{3}{n}(\psi_x\psi^*)-\frac{3(3-n)}{2n}(\psi\psi^*)_x$$

$$\frac{\delta\tilde{H}_{3,n}}{\delta\alpha_{n-3,n}}=\frac{\delta H_{3+n}}{\delta\alpha_{n-3,n}}+\frac{3}{n}\psi\psi^*.$$

To take into account $A_{i,i-1,n}=nD$ and $A_{n,n-2,n}=\frac{1}{2}(3n-n^2)D^2$, we obtain

$$nD\frac{\delta\tilde{H}_{3,n}}{\delta\alpha_{n-2,n}}=nD\frac{\delta H_{3+n}}{\delta\alpha_{n-2,n}}+3(\psi_x\psi^*)_x-\frac{3(3-n)}{2}(\psi\psi^*)_{xx} \tag{3.25}$$

$$nD\frac{\delta\tilde{H}_{3,n}}{\delta\alpha_{n-3,n}}=nD\frac{\delta H_{3+n}}{\delta\alpha_{n-3,n}}+3(\psi\psi^*)_x \tag{3.26}$$

and

$$\tfrac{1}{2}(3n-n^2)D^2\frac{\delta\tilde{H}_{3,n}}{\delta\alpha_{n-3,n}}=\tfrac{1}{2}(3n-n^2)D^2\frac{\delta H_{3+n}}{\delta\alpha_{n-3,n}}+\tfrac{3}{2}(3-n)(\psi\psi^*)_{xx}. \tag{3.27}$$

From (3.25) and (3.27) we get

$$nD\frac{\delta\tilde{H}_{3,n}}{\delta\alpha_{n-2,n}}+\tfrac{1}{2}(3n-n^2)D^2\frac{\delta\tilde{H}_{3,n}}{\delta\alpha_{n-3,n}}$$

$$=nD\frac{\delta H_{3+n}}{\delta\alpha_{n-2,n}}+\tfrac{1}{2}(3n-n^2)D^2\frac{\delta H_{3+n}}{\delta\alpha_{n-3,n}}+3(\psi_x\psi^*)_x. \tag{3.28}$$

Next we recall that lemma 2 yields that system (B) is equivalent to system (C). As (3.10) can be decomposed in the form

$$\alpha_{n_{t_3}} = A_{n,n}\frac{\delta H_{3+n}}{\delta \alpha_n} + \begin{pmatrix} 3(\psi_x\psi^*)_x \\ 3(\psi\psi^*)_x \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.29}$$

comparing (3.29) with (3.26)–(3.28), we can rewrite (3.29) as follows:

$$\alpha_{n_{t_3}} = A_{n,n}\frac{\delta \tilde{H}_{3,n}}{\delta \alpha_n}. \tag{3.30}$$

Finally from (3.21) we can easily write (1.2) as

$$\psi_{t_3} = \delta\tilde{H}_{3,n}/\delta\psi^* \tag{3.31}$$

$$\psi^*_{t_3} = -\delta\tilde{H}_{3,n}/\delta\psi \tag{3.32}$$

by using the definition of $\delta/\delta u$.

Therefore we can write (3.30)–(3.32) in the form of (3.23), i.e. system (B) can be written as the form of (3.23), and its conserved density is $\tilde{H}_{3,n}$ and the Poisson bracket is the same as (3.3). This ends the proof. □

4. Involutive conserved densities

We are now in a position to prove that the conserved densities $\tilde{H}_{2,n}$ and $\tilde{H}_{3,n}$ are involutive.

Theorem 4. If $\tilde{H}_{2,n}$ and $\tilde{H}_{3,n}$ are the conserved densities mentioned in theorems 2 and 3 respectively, then

$$\{\tilde{H}_{2,n}, \tilde{H}_{3,n}\} \sim 0. \tag{4.1}$$

Proof. To prove (4.1) is equivalent to proving that

$$\sum_{j=0}^{n}\left(\frac{\delta\tilde{H}_{3,n}}{\delta(\psi, \alpha_n, \psi^*)}\right)_j\left(B_{n,n}\frac{\delta\tilde{H}_{2,n}}{\delta(\psi, \alpha_n, \psi^*)}\right)_j \sim 0 \tag{4.2}$$

i.e.

$$\left(\frac{\delta\tilde{H}_{3,n}}{\delta(\psi, \alpha_n, \psi^*)}\right)^T B_{n,n}\frac{\delta\tilde{H}_{2,n}}{\delta(\psi, \alpha_n, \psi^*)} \sim 0 \tag{4.2a}$$

where $(\cdot)^T$ means the transposition.

Substituting the forms of $\tilde{H}_{2,n}$ and $\tilde{H}_{3,n}$ into (4.2), we obtain

$$\left[\begin{pmatrix} 0 \\ \frac{\delta H_{3+n}}{\delta \alpha_{0,n}} \\ \vdots \\ \frac{\delta H_{3+n}}{\delta \alpha_{n-n2,n}} \\ 0 \end{pmatrix} + \begin{pmatrix} -\psi^*_{t_3} \\ 0 \\ \vdots \\ 0 \\ \frac{3}{n}\psi\psi^* \\ \frac{3}{n}\psi_x\psi^* - \frac{3(3-n)}{2n}(\psi\psi^*)_x \\ \psi_{t_3} \end{pmatrix}\right]^T \begin{pmatrix} & & 1 \\ & A_{n,n} & \\ -1 & & \end{pmatrix} \left[\begin{pmatrix} 0 \\ \frac{\delta H_{2+n}}{\delta \alpha_{0,n}} \\ \vdots \\ \frac{\delta H_{2+n}}{\delta \alpha_{n-2,n}} \\ 0 \end{pmatrix} + \begin{pmatrix} -\psi^*_{t_2} \\ 0 \\ \vdots \\ 0 \\ \frac{2}{n}\psi\psi^* \\ \psi_{t_2} \end{pmatrix}\right] \sim 0.$$

Therefore, we only need to prove

$$\begin{aligned} &\frac{3}{n}\psi\psi^*\left(A_{n,n}\frac{\delta H_{2+n}}{\delta \alpha_n}\right)_{n-2} + \left[\frac{3}{n}(\psi_x\psi^*) - \frac{3}{2n}(3-n)(\psi\psi^*)_x\right]\left(A_{n,n}\frac{\delta H_{2+n}}{\delta \alpha_n}\right)_{n-1} \\ &\quad + 2\frac{\delta H_{3+n}}{\delta \alpha_{0,n}}(\psi\psi^*)_x + \left(\frac{\delta H_{3+n}}{\delta \alpha_n}\right)^T A_{n,n}\frac{\delta H_{2+n}}{\delta \alpha_n} \\ &\quad + \psi_{t_3}\psi^*_{t_2} - \psi^*_{t_3}\psi_{t_2} \sim 0. \end{aligned} \tag{4.3}$$

From the KP equation (1.3), (3.6), (3.22) and (3.29) we get

$$A_{n,n}\frac{\delta H_{2+n}}{\delta \alpha_{n-3,n}} = \alpha_{n-3,n_{t_2}} \tag{4.4}$$

$$A_{n,n}\frac{\delta H_{2+n}}{\delta \alpha_{n-2,n}} = \alpha_{n-2,n_{t_2}} \tag{4.5}$$

$$\frac{\delta H_{3+n}}{\delta \alpha_{0,n}} = \frac{1}{n}D^{-1}\alpha_{n-2,n_{t_3}} \tag{4.6}$$

$$\alpha_{n-2,n_{t_3}} = \tfrac{1}{4}\alpha_{n-2,n_{xxx}} + \frac{3}{n}\alpha_{n-2,n}\alpha_{n-2,n_x} + \tfrac{3}{4}D^{-1}\alpha_{n-2,n_{t_2t_2}} \tag{4.7}$$

$$\alpha_{n-3,n} = \tfrac{1}{2}D^{-1}\alpha_{n-2,n_{t_2}} + \frac{n-2}{2}\alpha_{n-2,n_x}. \tag{4.8}$$

Inserting (3.21), (4.4)-(4.8) and

$$\psi_{t_2} = \psi_{xx} + \frac{2}{n}\alpha_{n-2,n}\psi \tag{4.9a}$$

$$\psi^*_{t_2} = -\psi^*_{xx} - \frac{2}{n}\alpha_{n-2,n}\psi^* \tag{4.9b}$$

into (4.3), as a result of arduous calculations, we obtain
the left-hand side of (4.3)

$$\begin{aligned} &= \left(\frac{\delta H_{3+n}}{\delta \alpha_n}\right)^T A_{n,n}\frac{\delta H_{2+n}}{\delta \alpha_n} + \frac{3}{2n}[(\psi\psi^*)D^{-2}\alpha_{n-2,n_{t_2t_2}}]_x \\ &\quad + \frac{3(n-2)}{2n}(\psi\psi^*\alpha_{n-2,n_{t_2}})_x + \frac{3}{2n}[(\psi_x\psi^* - \psi\psi^*_x)D^{-1}\alpha_{n-2,n_{t_2}}]_x \\ &\quad - (\psi^*_{xx}\psi_{xx})_x - \frac{3}{n}[\alpha^2_{n-2,n}(\psi\psi^*)_x]_x + \frac{1}{2n}\alpha_{n-2,n_x}\psi\psi^*)_x \\ &\quad - \frac{2}{n}[\alpha_{n-2,n}(\psi\psi^*_{xx} + \psi^*\psi_{xx})]_x - \frac{1}{n}(\alpha_{n-2,n}\psi_x\psi^*_x)_x \sim 0 \end{aligned}$$

where we have used that [17, 19, 24]

$$\left(\frac{\delta H_{3+n}}{\delta \alpha_n}\right)^T A_{n,n} \frac{\delta H_{2+n}}{\delta \alpha_n} = \{H_{2+n}, H_{3+n}\} \sim 0.$$

This ends the proof of this theorem. □

5. Remarks

It is natural to generalize the constraints used in this paper to the whole KP hierarchy, i.e. we can consider the system

$$\psi_{tm} = \mathrm{L}_+^m \psi$$

$$\psi_{tm}^* = -\mathrm{L}_+^{m*} \psi^* \qquad m, n = 1, 2, \ldots$$

$$K_{n-1} = (\psi\psi^*)_x$$

where $\mathrm{L}_+^{m^*}$ is the differential operator formally adjoint to the operator L_+^m.

We would like to indicate that a common infinite set of conserved densities can be obtained for this system by using the Lie algebraic framework [12], and that similar results for other 2 + 1-dimensional nonlinear equations, such as the Davey–Stewartson hierarchy, etc. will be left to the sequel.

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References

[1] Cao Cewen 1989 *Scientia Sinica* 7 701
[2] Cao Cewen and Gen Xianguo 1990 *Nonlinear Physics—Proceeding (Shanghai, 1989) (Research Reports in Physics 68)* (Heidelberg: Springer)
[3] Moser J 1980 Various aspects of integrable Hamiltonian systems in *Progress in Mathematics* vol 8 (Basel: Birkhäuser) p 233
[4] Yunbo Zeng and Yishen Li 1989 *J. Math. Phys.* **30** 1679
[5] Yunbo Zeng and Yishen Li 1989 *J. Phys. A: Math. Gen.* **23** L89
[6] Yunbo Zeng and Yishen Li 1990 *J. Math. Phys.* **31**
[7] Yunbo Zeng and Yishen Li 1991 A unified approach to the Liouville integrability of finite-dimensional Hamiltonian systems reduced from zero-curvature equations associated with $\widetilde{sl}$ (2) *Preprint* USTC
[8] Yunbo Zeng 1991 An approach to the deduction of the finite-dimensional integrability from the inifinite-dimensional integrability *Phys. Lett.* **160A** 541
[9] Yi Cheng and Yishen Li 1991 *Phys. Lett.* **157A** 22
[10] Konopelchenko B G, Strampp W and Sidorenko J 1991 *Phys. Lett.* **157A** 17
[11] Yi Cheng 1992 Constraints of the (2 + 1)-dimensional integrable soliton systems *J. Phys. A: Math. Gen.* **25** 419

[12] Bing Xu (1+1)-dimensional integrable Hamiltonian systems reduced from symmetry constraints of the KP hierarchy *Preprint* USTC (submitted to *Inverse Problems*)
[13] Jimbo M and Miwa T 1983 *Publ. RIMS, Kyoto Univ.* **19** 943.
[14] Date E, Kashiwara M, Jimbo M and Miwa T 1983 *Non-linear Integrable Systems—Classical Theory and Quantum Theory* ed M Jimbo and T Miwa (Singapore: World Scientific)
[15] Segal G and Wilson G 1985 Publ. IHES
[16] Bing Xu 1989 The theory of the Burchall-Chaudry matrix and its application in KdV and Boussinesq type equations *Preprint* USTC
[17] Adler M 1979 *Invent. Math.* **50** 219
[18] Mumford S 1984 *Tata Lectures on Theta II* (Basel: Birkhäuser)
[19] Strampp W and Oevel W 1990 *Lett. Math. Phys.* **20** 195
[20] Fokas A S 1987 *Stud. Appl. Math.* **77** 253
[21] Santini P M and Fokas A S 1988 *Commun. Math. Phys.* **115** 375
[22] Konopelchenko B G 1988 *Inverse Problems* **4** 75
[23] Yajima N and Oikawa M 1976 *Prog. Theor. Phys.* **56** 1719
[24] Manin Y I 1979 *J. Soviet Math.* **11** 1